Tau functions from matrix models in enumerative geometry and isomonodromic deformations

Tau Functions of Integrable Systems and Theirs Applications
BIRS, Banff, 4 September 2018

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Overview

Some tau functions from enumerative geometry/2D topological field theories have an isomonodromic interpretation.

Examples:

- intersection theory on the moduli space of Riemann surfaces (Kontsevich–Witten tau function) [Bertola and Cafasso, CMP 2017]
- open version (Kontsevich–Penner tau function) [Bertola and R, arXiv:1711.03360]
- r–spin version [in progress]
- Brezin–Gross–Witten tau function [in progress]
- stationary sector of the Gromov–Witten theory of $\mathbb{P}^1$ [in progress]

Applications:

- rigorous asymptotic study of large N matrix integrals
- explicit generating functions for correlators
- derivation of Virasoro constraints

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Plan of the talk

- Detailed exposition of the result in the case of the Kontsevich–Witten tau function
- Outline of the other cases

The Kontsevich–Witten tau function

Let $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{\mathscr{M}}_{g,n}} \psi_1^{d_1} \wedge \cdots \wedge \psi_n^{d_n}$ be the Witten intersection numbers ($d_1, ..., d_n \geq 0$, $d_1 + \cdots + d_n = 3g - 3 + n$). Form the generating function

$$F(t_1, t_3, ...) = \sum_{n \geq 1} \sum_{d_1, \ldots, d_n \geq 0} \frac{t_{2d_1+1} \cdots t_{2d_n+1}}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \frac{t_1^3}{6} + \frac{t_3}{24} + \frac{t_1 t_5}{24} + \frac{t_3^2}{24} + \frac{t_1^2 t_7}{48} + \cdots$$

Witten, Kontsevich, Dijkgraaf, Verlinde, Verlinde,...1991-1992

$\tau^{KW}(t_1, t_3, ...) := \exp F(t_1, t_3, ...)$ is a KdV tau function, uniquely selected by the string equation $L_{-1} \tau^{KW} = 0$,

$$L_{-1} = -\frac{\partial}{\partial t_1} + \sum_{a \geq 1,\ a \text{ odd}} t_{a+2} \frac{\partial}{\partial t_a} + \frac{t_1^2}{2}.$$

Equivalently: $\tau^{KW}(t_1, t_3, ...)$ satisfies $L_k \tau^{KW} = 0$ for $k \geq -1$. The operators

$$L_k = \sum_{a \geq 1,\ a \text{ odd}} \frac{(a+2k)!!}{(a-2)!!} (t_a - \delta_{a,3}) \frac{\partial}{\partial t_{a+2k}} + \frac{1}{2} \sum_{\substack{a,b \geq 1,\ a,b \text{ odd} \\ a+b=2k}} a!! b!! \frac{\partial^2}{\partial t_a \partial t_b} + \frac{t_1^2}{2} \delta_{k,-1} + \frac{1}{8} \delta_{k,0}$$

are called Virasoro operators. They commute as $[L_k, L_l] = (k-l) L_{k+l}$ for $k, l \geq -1$.

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The Kontsevich matrix integral

Fix $N \geq 1$ and take the ratio of determinants

$$\tau_N(T_1, \ldots, T_N) := \left. \frac{\det[\phi_j(\lambda_k)]_{j,k=1}^N}{\det[\lambda_k^{\frac{j-1}{2}}]_{j,k=1}^N} \right|_{T_\ell := \frac{1}{\ell}(\lambda_1^{-\ell/2} + \cdots + \lambda_N^{-\ell/2})}$$

where $\phi_j(\lambda) = \lambda^{\frac{j-1}{2}}(1 + \mathcal{O}(\lambda^{-3/2})) \in \lambda^{\frac{j-1}{2}}\mathbb{C}[\![\lambda^{-\frac{3}{2}}]\!]$ are defined by

$$\left(-\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{j-1}\mathrm{Ai}(\lambda) \sim \frac{\exp\left(-\frac{2}{3}\lambda^{3/2}\right)}{2\sqrt{\pi}\lambda^{1/4}}\phi_j(\lambda), \quad \lambda \to +\infty \quad (j \geq 1).$$

Equivalently, $\tau_N(T_1, \ldots, T_N)$ is the asymptotic expansion of the Kontsevich matrix integral

$$Z_N(\Lambda) := \int_{H_N} \exp \operatorname{tr}\left(\mathrm{i}\frac{X^3}{3} - \Lambda^{1/2}X^2\right)\mathrm{d}X \Big/ \int_{H_N} \exp \operatorname{tr}\left(-\Lambda^{1/2}X^2\right)\mathrm{d}X$$

for positive large $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$, expressed in the Miwa variables $T_\ell = \frac{1}{\ell}\operatorname{tr}\Lambda^{-\ell/2}$. The power series $\tau_N(T_1, \ldots, T_N) \in \mathbb{C}[\![T_1, \ldots, T_N]\!]$ have a stable limit $\tau(T_1, T_3, \ldots)$ when $N \to \infty$ which is by construction a KdV tau function.

Kontsevich, 1992

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The Airy Stokes' phenomenon

To describe the isomonodromic formulation of the Kontsevich–Witten tau function we start by the Airy ODE $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_0(\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} \Psi_0(\lambda)$. Its Stokes' phenomenon at $\lambda = \infty$ can be summarized in the Riemann–Hilbert problem below.

$$\Psi_0(\lambda+) = \Psi_0(\lambda-)M$$
$$\Psi_0(\lambda) \sim \lambda^S G(\mathbf{1} + \mathcal{O}(\lambda^{-\frac{1}{2}}))\mathrm{e}^{\vartheta(\lambda)},\ \lambda \to \infty$$
$\Psi_0(\lambda)$ analytic, analytically invertible

$$S = -\frac{1}{4}\sigma_3$$
$$G = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
$$\vartheta(\lambda) = \frac{2}{3}\lambda^{\frac{3}{2}}\sigma_3$$

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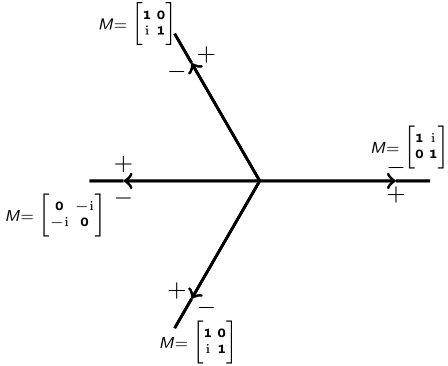

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Schlesinger transformations (à la Bertola–Cafasso)

Fix $N \geq 1$ and points $\lambda_1, ..., \lambda_N$ and introduce the matrix

$$D_N(\lambda) = D_N(\lambda; \lambda_1, ..., \lambda_N) := \prod_{j=1}^{N} \mathrm{diag}(\sqrt{\lambda_j} - \sqrt{\lambda}, \sqrt{\lambda_j} + \sqrt{\lambda}).$$

Find $\Psi_N(\lambda) = \Psi_N(\lambda; \lambda_1, ..., \lambda_N)$ such that

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$\Psi_N(\lambda) D_n(\lambda)$ analytic and analytically invertible.

If solvable, by Liouville Theorem we get an isomonodromic system:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_N(\lambda; \lambda_*) = A_N(\lambda; \lambda_*)\Psi_N(\lambda; \lambda_*), \quad \frac{\mathrm{d}}{\mathrm{d}\lambda_j}\Psi_N(\lambda; \lambda_*) = \Omega_{j,N}(\lambda; \lambda_*)\Psi_N(\lambda; \lambda_*)\ (1 \leq j \leq N).$$

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The limit $N \to \infty$

The tau function depends only on $D_N^{-1} M D_N$ (Bertola–Malgrange form). Then we can pass to the limit $N \to \infty$ by replacing

$$D_N^{-1}(\lambda, \lambda_1, ..., \lambda_N) \mapsto \exp \sum_{\ell \geq 1,\ \ell \text{ odd}} \sigma_3 T_\ell \lambda^{\ell/2}$$

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Therefore we consider the following Riemann–Hilbert problem:

$$\Psi(\lambda+; T_*) = \Psi(\lambda-; T_*)M, \quad \Psi(\lambda; T_*) \sim \lambda^S G(\mathbf{1} + \mathcal{O}(\lambda^{-1/2}))\mathrm{e}^{\Theta(\lambda; T_*)},\ \lambda \to \infty$$

$$\Psi(\lambda; T_*) \text{ analytic, analytically invertible, } \Theta(\lambda; T_*) = \exp \sum_{\ell \geq 1,\ \ell \text{ odd}} \sigma_3 (T_\ell + \frac{2}{3}\delta_{\ell,3})\lambda^{\ell/2}$$

Again we have an isomonodromic system (A, Ω_ℓ polynomials in λ)

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One–point function

The isomonodromic tau function is defined by the Jimbo–Miwa–Ueno formula:

$$\frac{\partial}{\partial T_{2d+1}}\log\tau(T_*) = -\operatorname*{res}_{\lambda=\infty}\mathrm{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\frac{\partial\Theta}{\partial T_{2d+1}}\right)\mathrm{d}\lambda = -\operatorname*{res}_{\lambda=\infty}\mathrm{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\sigma_3\right)\lambda^{\frac{2d+1}{2}}\mathrm{d}\lambda$$

Therefore we can compute the one–point function as follows:

$$-\sum_{d\geq 0}\frac{(2d+1)!!}{2^{-\frac{2d+1}{3}}\lambda^{d+1}}\langle\tau_d\rangle = -\sum_{d\geq 0}\frac{(2d+1)!!}{2^{-\frac{2d+1}{3}}\lambda^{d+1}}\left.\frac{\partial\log\tau(t_*)}{\partial t_{2d+1}}\right|_{t_*=0} =$$

$$=\sum_{d\geq 0}\frac{1}{\lambda^{d+1}}\left.\frac{\partial\log\tau(T_*)}{\partial T_{2d+1}}\right|_{T_*=0} = -\sum_{d\geq 0}\frac{1}{\lambda^{d+1}}\operatorname*{res}_{\mu=\infty}\mathrm{tr}\left(\mu^{1/2}\Psi_0^{-1}(\mu)\frac{\mathrm{d}\Psi_0(\mu)}{\mathrm{d}\mu}\sigma_3\right)\mu^d\mathrm{d}\mu =$$

$$=\mathrm{tr}\left(\lambda^{1/2}\Psi_0^{-1}(\lambda)\frac{\mathrm{d}\Psi_0(\lambda)}{\mathrm{d}\lambda}\sigma_3\right)$$

and $\Psi(\lambda; T_*=0)=\Psi_0(\lambda)$ is known (Airy functions) therefore (after a simple Laplace–Borel transform) we obtain the well known formula (Itzykson and Zuber, 1991)

$$\sum_{d\geq 0}\langle\tau_{d-2}\rangle X^d = \exp\frac{X^3}{24},\ \text{i.e. } \langle\tau_{3g-2}\rangle = \frac{1}{24^g g!}.$$

One–point function

The isomonodromic tau function is defined by the Jimbo–Miwa–Ueno formula:

$$\frac{\partial}{\partial T_{2d+1}} \log \tau(T_*) = -\operatorname*{res}_{\lambda=\infty} \operatorname{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\frac{\partial\Theta}{\partial T_{2d+1}}\right)\mathrm{d}\lambda = -\operatorname*{res}_{\lambda=\infty} \operatorname{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\sigma_3\right)\lambda^{\frac{2d+1}{2}}\mathrm{d}\lambda$$

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n–point function

Following a similar strategy, from the Jimbo–Miwa–Ueno formula we can compute inductively the n–point functions:

Bertola, Dubrovin and Yang, 2015

Let

$$A(\lambda) := \begin{bmatrix} -\frac{1}{2}\sum_{g\geq 1}\frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-3g+2} & -\sum_{g\geq 0}\frac{(6g-1)!!}{24^g g!}\lambda^{-3g} \\ \sum_{g\geq 0}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g g!}\lambda^{-3g+1} & \frac{1}{2}\sum_{g\geq 1}\frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-3g+2} \end{bmatrix}.$$

Then

$$\sum_{d_1,d_2=0}^{\infty}\frac{(2d_1+1)!!(2d_2+1)!!}{\lambda_1^{d_1+1}\lambda_2^{d_2+1}}\langle\tau_{d_1}\tau_{d_2}\rangle = \mathrm{tr}\,\frac{A(\lambda_1)A(\lambda_2)}{(\lambda_1-\lambda_2)^2} - \frac{\lambda_1+\lambda_2}{(\lambda_1-\lambda_2)^2}$$

$$\sum_{d_1,\dots,d_n=0}^{\infty}\frac{(2d_1+1)!!\cdots(2d_n+1)!!}{\lambda_1^{d_1+1}\cdots\lambda_n^{d_n+1}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle = -\frac{1}{n}\sum_{\sigma\in S_n}\frac{\mathrm{tr}\left(A\left(\lambda_{\sigma_1}\right)\cdots A\left(\lambda_{\sigma_n}\right)\right)}{\prod_{j\in\frac{\mathbb{Z}}{n\mathbb{Z}}}\left(\lambda_{\sigma_j}-\lambda_{\sigma_{j+1}}\right)}.$$

Virasoro constraints

All the Virasoro constraints follow from the fact that a total differential is residueless:

$$\operatorname*{res}_{\lambda=\infty} \operatorname{tr} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\lambda^{\frac{2(d+k+1)+1}{2}}\sigma_3\right)\mathrm{d}\lambda = 0 \Rightarrow \frac{\partial}{\partial T_{2d+1}}(L_k\tau) = 0,\ \ k \geq -1.$$

E.g. let us derive the string equation $L_{-1}\tau = 0$: we need one preliminary observation

$$\Omega_\ell = \left(\frac{\partial\Psi}{\partial T_\ell}\Psi^{-1}\right)_+ = \left(\Psi\frac{\partial\Theta}{\partial T_\ell}\Psi^{-1}\right)_+ = \left(\Psi\sigma_3\Psi^{-1}\lambda^{\ell/2}\right)_+$$

$$A = \left(\frac{\partial\Psi}{\partial\lambda}\Psi^{-1}\right)_+ = \left(\Psi\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda}\Psi^{-1}\right)_+ = \sum_{\ell\geq 1,\ \ell\ \mathrm{odd}}\left(\Psi\frac{\ell}{2}\widetilde{T}_\ell\lambda^{\frac{\ell}{2}-1}\sigma_3\Psi^{-1}\right)_+ = \sum_{\ell\geq 1,\ell\ \mathrm{odd}}\frac{\ell}{2}\widetilde{T}_\ell\Omega_{\ell-2}$$

where $\widetilde{T}_\ell = T_\ell + \frac{2}{3}\delta_{\ell,3}$. Hence

$$0 = -\operatorname*{res}_{\lambda=\infty}\operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\Psi^{-1}A\Psi\lambda^{\frac{2d+1}{2}}\sigma_3\right)\mathrm{d}\lambda = -\operatorname*{res}_{\lambda=\infty}\operatorname{tr}\left(\Psi^{-1}\frac{\mathrm{d}A}{\mathrm{d}\lambda}\Psi\sigma_3\lambda^{\frac{2d+1}{2}} + \frac{2d+1}{2}\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\sigma_3\lambda^{\frac{2d-1}{2}}\right)\mathrm{d}\lambda =$$

$$= -\sum_{\ell\geq 1,\ \mathrm{odd}}\frac{\ell}{2}\widetilde{T}_\ell\operatorname*{res}_{\lambda=\infty}\operatorname{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Omega_{\ell-2}}{\mathrm{d}\lambda}\Psi\sigma_3\lambda^{\frac{2d+1}{2}}\right)\mathrm{d}\lambda - \frac{2d+1}{2}\frac{\partial\log\tau}{\partial T_{2d-1}} + \frac{1}{2}\delta_{d,0}T_1 =$$

$$= \sum_{\ell\geq 1,\ \mathrm{odd}}\frac{\ell}{2}\widetilde{T}_\ell\frac{\partial^2\log\tau}{\partial T_{\ell-2}\partial T_{2d+1}} + \frac{2d+1}{2}\frac{\partial\log\tau}{\partial T_{2d-1}} + \frac{1}{2}\delta_{d,0}T_1 = \frac{\partial}{\partial T_{2d+1}}\left(\sum_{\ell\geq 1,\ \mathrm{odd}}\frac{\ell}{2}\widetilde{T}_\ell\frac{\partial\log\tau}{\partial T_{\ell-2}} + \frac{T_1^2}{4}\right) = \frac{\partial}{\partial T_{2d+1}}\left(\frac{L_{-1}\tau}{\tau}\right)$$

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$$=-\sum_{\ell\geq 1,\ \mathrm{odd}}\frac{\ell}{2}\widetilde{T}_\ell\operatorname*{res}_{\lambda=\infty}\mathrm{tr}\left(\Psi^{-1}\frac{\mathrm{d}\Omega_{\ell-2}}{\mathrm{d}\lambda}\Psi\sigma_3\lambda^{\frac{2d+1}{2}}\right)\mathrm{d}\lambda-\frac{2d+1}{2}\frac{\partial\log\tau}{\partial T_{2d-1}}+\frac{1}{2}\delta_{d,0}T_1=$$

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The Kontsevich–Penner tau function

The Kontsevich–Penner matrix integral is ($\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$, Q a parameter)

$$\int_{H_N} \frac{\exp \mathrm{tr}\left(\mathrm{i}\frac{X^3}{3} - \Lambda^{1/2}X^2\right)}{\det(\mathbf{1} - \mathrm{i}X\Lambda^{-1/2})^Q}\mathrm{d}X \Bigg/ \int_{H_N} \exp \mathrm{tr}(-\Lambda^{1/2}X^2)\mathrm{d}X.$$

As in the Kontsevich case, sending $N \to \infty$ one can build the Kontsevich–Penner tau function, which is a formal KP tau function $\tau(T_1, T_2, ...; Q)$ in the Miwa times

$$T_\ell(\Lambda) = \frac{1}{\ell}\,\mathrm{tr}\,\Lambda^{-\ell/2}.$$

Conjecture (Pandharipande–Solomon–Tessler, Alexandrov–Buryak–Tessler, Safnuk,...)

Let $F(t_1, t_2, ...; Q)$ be the generating function for the (refined) open intersection numbers

$$F(t_1, t_2, ...; Q) := \sum_{n\geq 1}\sum_{b\geq 0} Q^b \sum_{r_1,\dots,r_n\geq 0} \frac{t_{r_1+1}\cdots t_{r_n+1}}{n!}\langle \tau_{\frac{r_1}{2}}\cdots\tau_{\frac{r_n}{2}}\rangle_b =$$

$$= \frac{t_1^3}{6} + \frac{t_3}{24}(1+12Q^2) + Qt_1t_2 + \frac{t_2t_4}{2}Q^2 + \frac{t_3^2}{24}(1+12Q^2) + Qt_1^2t_4 + ...$$

Then $F(t_1, t_2, ...; Q) = \log \tau(T_1, T_2, ...)$ with $T_k = (-1)^k \frac{2^{k/3}}{k!!} t_k$.

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The Kontsevich–Penner tau function as an isomonodromic tau function

As done by Bertola and Cafasso for the Kontsevich tau function, we want to identify the Kontsevich–Penner tau function with an isomonodromic tau function. This is done in the same way as explained above, but starting from the following variation of the Airy equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_0(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q & \lambda & 0 \end{bmatrix} \Psi_0(\lambda)$$

Bertola and R, 2017

The isomonodromic tau function associated to the isomonodromic system obtained as above by Schlesinger transformations of ODE above at the points $\lambda_1, ..., \lambda_N$ coincides with the Kontsevich–Penner matrix integral.

One can pass (formally) to the limit $N \to \infty$ and apply the same considerations above to identify the Kontsevich–Penner tau function with the isomonodromic tau function of a 3×3 system. Therefore generating functions for n–point functions and Virasoro constraints can be computed in the same way.

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One–point function (open case)

By the same strategy explained above we are able to compute the n–point functions.

Bertola and R, 2017

$$\sum_{r\geq 0}\langle \tau_{\frac{r}{2}-2}\rangle X^{\frac{r}{2}} = \mathrm{e}^{\frac{X^3}{6}}\left({}_2F_2\left(\left.\begin{matrix}\frac{1}{2}-Q & \frac{1}{2}+Q\\ \frac{1}{2} & \frac{1}{2}\end{matrix}\right| -\frac{X^3}{8}\right) + Q\,X^{\frac{3}{2}}\,{}_2F_2\left(\left.\begin{matrix}1-Q & 1+Q\\ 1 & \frac{3}{2}\end{matrix}\right| -\frac{X^3}{8}\right)\right) =$$

$$= 1 + QX^{\frac{3}{2}} + \frac{1+12Q^2}{24}X^3 + \frac{Q+Q^3}{12}X^{\frac{9}{2}} + \frac{1+56Q^2+16Q^4}{1152}X^6 + \ldots$$

For $Q=0$ it reduces correctly to the closed case giving $\exp\frac{X^3}{24}$.

An equivalent alternative expression for the same one–point function:

$$\sum_{r\geq 0}\langle \tau_{\frac{r}{2}-2}\rangle X^{\frac{r}{2}} = \mathrm{e}^{\frac{X^3}{24}}\sum_{j\geq 0}\frac{A_j(Q)}{(j-1)!!}X^{\frac{3j}{2}}, \quad \left(\frac{2+X}{2-X}\right)^Q = \sum_{j\geq 0}A_j(Q)X^j.$$

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n–point function (open case)

Introduce $P^k_{a,b}(Q)$ (polynomials in Q, $a, b = 0, \pm 1$, $k = 0, 1, 2, ...$)

$$\sum_{m\geq 0} \frac{\Gamma\left(\frac{a-b+1}{2}\right)}{\Gamma\left(\frac{a-b+1+6m}{2}\right)} Z^m P^{2m}_{a,b}(Q) = \mathrm{e}^{\frac{Z}{3}}\, {}_2F_2\left(\begin{matrix} \frac{1-a-b-2Q}{2} & \frac{1+a+b+2Q}{2} \\ \frac{1}{2} & \frac{1+a-b}{2} \end{matrix} \middle| -\frac{Z}{4}\right)$$

$$\sum_{m\geq 0} \frac{\Gamma\left(\frac{a-b+2}{2}\right)}{\Gamma\left(\frac{a-b+4+6m}{2}\right)} Z^m P^{2m+1}_{a,b}(Q) = -\frac{2Q+a+b}{2} \mathrm{e}^{\frac{Z}{3}}\, {}_2F_2\left(\begin{matrix} \frac{2-a-b-2Q}{2} & \frac{2+a+b+2Q}{2} \\ \frac{3}{2} & \frac{2+a-b}{2} \end{matrix} \middle| -\frac{Z}{4}\right)$$

and the matrix $A(\lambda)$

$$A(\lambda) := \sum_{k\geq 0} \begin{bmatrix} Q\, P^k_{1,-1}(Q)\lambda^{-\frac{3k+2}{2}} & P^k_{-1,-1}(Q)\lambda^{-\frac{3k}{2}} & P^k_{0,-1}(Q)\lambda^{-\frac{3k+1}{2}} \\ Q\, P^k_{1,0}(Q)\lambda^{-\frac{3k+1}{2}} & P^k_{-1,0}(Q)\lambda^{-\frac{3k-1}{2}} & P^k_{0,0}(Q)\lambda^{-\frac{3k}{2}} \\ Q\, P^k_{1,1}(Q)\lambda^{-\frac{3k}{2}} & P^k_{-1,1}(Q)\lambda^{-\frac{3k-2}{2}} & P^k_{0,1}(Q)\lambda^{-\frac{3k-1}{2}} \end{bmatrix}.$$

Bertola and R, 2017

For $n \geq 2$ we have

$$\sum_{r_1,..,r_n\geq 0} \left\langle \prod_{i=1}^n \frac{(-1)^{r_i} r_i!!}{2^{\frac{r_i}{3}} \lambda_i^{\frac{r_i+1}{2}}} \tau_{\frac{r_i-1}{2}} \right\rangle = -\frac{1}{n} \sum_{i\in S_n} \mathrm{tr}\, \frac{A(\lambda_{i_1})\cdots A(\lambda_{i_n})}{(\lambda_{i_1}-\lambda_{i_2})\cdots(\lambda_{i_n}-\lambda_{i_1})} - \frac{\delta_{n,2}}{\left(\lambda_1^{\frac{1}{2}} - \lambda_2^{\frac{1}{2}}\right)^2}.$$

Open Virasoro constraints

The Kontsevich–Penner tau function satisfies

$$L_k \tau = 0, \quad k \geq -1$$

where

$$L_k(Q) = \sum_{a\geq 1} \frac{a}{2}\left(T_a + \frac{3}{2}\delta_{a,3}\right)\frac{\partial}{\partial T_{a+2k}} + \frac{1}{4}\sum_{a,b\geq 1,\ a+b=2k} \frac{\partial^2}{\partial T_a \partial T_b} + $$

$$+\frac{3}{2}Q\frac{\partial}{\partial T_{2k}} + \left(\frac{T_1^2}{4} + QT_2\right)\delta_{k,-1} + \left(\frac{1}{16} + \frac{3}{4}Q^2\right)\delta_{k,0}$$

Such Virasoro constraints can be obtained by the isomonodromic method as explained before and they coincide with those computed by Alexandrov (2016).

r–spin intersection numbers

More generally one can consider a model of the form ($\Lambda = Y^r$)

$$\int_{\mathcal{H}_N(\gamma)} \exp\operatorname{tr} \frac{\left(\frac{(X+Y)^{r+\mathbf{1}} - Y^{r+\mathbf{1}}}{r+\mathbf{1}} - XY^r\right)}{\det(XY^{-1} + \mathbf{1})^Q} \mathrm{d}X \Big/ \int_{\mathrm{i}\mathcal{H}_N} \exp\operatorname{tr}\left(\frac{1}{2}\sum_{a=\mathbf{0}}^{r-\mathbf{1}} XY^a XY^{r-1-a}\right) \mathrm{d}X.$$

$r = 2 \Rightarrow$ Kontsevich–Penner model; $Q = 0 \Rightarrow r$–spin (closed) intersection numbers.

R, in progress

This matrix model coincides with the isomonodromic tau function of the $(r+1)\times(r+1)$ isomonodromic system which is built as above by Schlesinger transformations at the N points $\lambda_j = y_j^r$ of the ODE

$$\frac{\mathrm{d}\Psi_0(\lambda)}{\mathrm{d}\lambda} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & 1 \\ Q & \lambda & \cdots & 0 \end{bmatrix} \Psi_0(\lambda).$$

The same arguments about the formal limit $N \to +\infty$ can be applied to this case $\Rightarrow$ we can compute open r–spin intersection numbers and the open r-spin Virasoro constraints (in progress).

The stationary sector of the GW theory of $\mathbb{P}^1$

The stationary Gromov–Witten invariants of $\mathbb{P}^1$ are the rational numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,\beta} := \int_{\overline{\mathcal{M}_{g,n}(\mathbb{P}^1,\beta)}} \bigwedge_{i=1}^{n} \mathrm{ev}_i^*(\omega) \bigwedge_{i=1}^{n} c_1\left(\mathcal{L}_i\right)^{d_i}$$

with nonvanishing condition $d_1 + \ldots + d_n = 2g - 2 + 2\beta$, $\beta \in H_2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$, $\omega \in H^2(\mathbb{P}^1, \mathbb{C})$, $\int_{\mathbb{P}^1} \omega = 1$.

Recently Dubrovin and Yang have found explicit formulae for n–point functions; such formulae have been later proved with Zagier (an independent proof which makes use of the Topological Recursion was given by Marchal).

Goal: identify such formulae with the isomonodromic approach; do they come from a matrix model?

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The Riemann–Hilbert problem for the stationary sector of the GW theory of $\mathbb{P}^1$: the bare problem

Let us denote by $J_\nu(x)$ the standard Bessel function;

$$J_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n}.$$

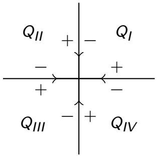
Introduce the piecewise analytic matrix

$$a_I(\lambda) := \sqrt{2\pi s}J_{\lambda-\frac{1}{2}}(2s), \quad b_I(\lambda) := \sqrt{\frac{\pi s}{2}}\left(-\frac{2\mathrm{i}\mathrm{e}^{2\pi\mathrm{i}\lambda}}{\mathrm{e}^{2\pi\mathrm{i}\lambda}+1}J_{\lambda+\frac{1}{2}}(2s) + \frac{1}{\cos(\pi\lambda)}J_{-\lambda-\frac{1}{2}}(2s)\right),$$

$$a_{IV}(\lambda) := \sqrt{2\pi s}J_{\lambda-\frac{1}{2}}(2s), \quad b_{II}(\lambda) := \sqrt{\frac{\pi s}{2}}\left(\frac{2\mathrm{i}}{\mathrm{e}^{2\pi\mathrm{i}\lambda}+1}J_{\lambda+\frac{1}{2}}(2s) + \frac{1}{\cos(\pi\lambda)}J_{-\lambda-\frac{1}{2}}(2s)\right),$$

$$a_{II}(\lambda) := \mathrm{e}^{-\mathrm{i}\pi\lambda}b_{IV}(-\lambda), \; b_{II}(\lambda) := \mathrm{e}^{\mathrm{i}\pi\lambda}a_{IV}(-\lambda), \; a_{III}(\lambda) := \mathrm{e}^{\mathrm{i}\pi\lambda}b_I(-\lambda), \; b_{III}(\lambda) := \mathrm{e}^{-\mathrm{i}\pi\lambda}a_I(-\lambda).$$

$$\Psi(\lambda)\Big|_{\lambda\in Q_\ell} := \begin{bmatrix} a_\ell(\lambda;s) & b_\ell(\lambda-1;s) \\ a_\ell(\lambda+1;s) & b_\ell(\lambda;s) \end{bmatrix}, \quad \ell\in\{I,II,III,IV\}$$



Bertola and R, in progress

$\Psi(\lambda)$ is analytic and analytically invertible

$$\Psi(\lambda) \sim \left(1+\mathcal{O}\left(\lambda^{-1}\right)\right)\left(\frac{\mathrm{e}s}{\lambda}\right)^{\lambda\sigma_3}$$

$$\Psi(\lambda-1) = \begin{bmatrix} \frac{\lambda}{s}-\frac{1}{2s} & -1 \\ 1 & 0 \end{bmatrix}\Psi(\lambda)$$

$$\det\Psi(\lambda)\equiv 1$$

The Riemann–Hilbert problem for the stationary sector of the GW theory of $\mathbb{P}^1$: dressing and the tau function

Dress the jump matrices by

$$\exp \xi(\lambda; t_*), \quad \xi(\lambda; t_*) := \frac{1}{2}\sigma_3 \sum_{k\geq 1} t_k \lambda^k$$

and let $\Psi(\lambda; t_*) = \Gamma(\lambda; t_*) \left(\frac{\mathrm{e}s}{\lambda}\right)^{\lambda\sigma_3}$ the solution of the dressed RHp.

The matrix $\widehat{\Psi}(\lambda; t_*) := \Psi(\lambda; t_*)\mathrm{e}^{\xi(\lambda;t_*)}$ satisfy deformation equations

$$\partial_{t_j} \widehat{\Psi}(\lambda; t_*) = \Omega_j(\lambda; t_*)\widehat{\Psi}(\lambda; t_*), \quad \Omega_j(\lambda; t_*) = \frac{1}{2}\left(\Gamma(\lambda; t_*)\sigma_3\Gamma^{-1}(\lambda; t_*)\lambda^j\right)_+$$

and the Jimbo–Miwa–Ueno one–form is closed:

$$\omega\left(\partial_{t_j}\right) := -\operatorname*{res}_{\lambda=\infty} \operatorname{tr} \frac{1}{2}\left(\Gamma^{-1}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\lambda}\,\sigma_3\right)\lambda^j \mathrm{d}\lambda, \quad \partial_{t_a}\omega\left(\partial_{t_b}\right) = \partial_{t_b}\omega\left(\partial_{t_a}\right).$$

Therefore introduce

$$\partial_{t_j} \log \tau(t_*) = \omega\left(\partial_{t_j}\right).$$

Proposition (Bertola and R, in progress)

By construction, the logarithmic derivatives of $\tau(t_*)$ evaluated at $t_* = 0$ coincide with Dubrovin, Yang and Zagier's formulae for the stationary GW invariants of $\mathbb{P}^1$.

The matrix model for the stationary sector of the GW theory of $\mathbb{P}^1$: two Bessel matrix functions

The following model was conjectured to yield stationary GW invariants of $\mathbb{P}^1$ (Aganagic, Dijkgraaf, Klemm, Marino and Vafa, 2006):

$$f(\lambda) := \int_{-\infty}^{+\infty} \exp\left(\lambda x - \mathrm{e}^x - \mathrm{e}^{-x}\right) \mathrm{d}x = \int_0^{+\infty} \exp\left(\lambda \log t - t - t^{-1}\right) \frac{\mathrm{d}t}{t} = 2K_{-\lambda}(2).$$

$f(\lambda)$ admits two *distinct* matrix versions $\lambda \to \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$:

$$\int_{H_N} \exp \mathrm{tr}\left(\Lambda X - \mathrm{e}^X - \mathrm{e}^{-X}\right) \mathrm{d}X = c_N \frac{\det\left[\partial_\lambda^{b-1} f(\lambda_a)\right]_{a,b=1}^N}{\Delta(\lambda_1, \dots, \lambda_N)}$$

$$\int_{H_N^+} \exp \mathrm{tr}\left(\Lambda \log T - T - T^{-1}\right) \frac{\mathrm{d}T}{\det T} = c_N' \frac{\det\left[f(\lambda_a + b - 1)\right]_{a,b=1}^N}{\Delta(\lambda_1, \dots, \lambda_N)}$$

The matrix model we obtain from Schlesinger transformations is

$$\tau_N(\lambda_1, \dots, \lambda_N) = \frac{\det\left[\Psi_{22}\left(\lambda_a + b - 1\right)\left(\frac{\mathrm{e}s}{\lambda_a}\right)^{\lambda_a}\right]_{a,b=1}^N}{\Delta(\lambda_1, \dots, \lambda_N)}.$$

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Open problems

- Understand better the relation with the matrix models
- Add descendants of the identity operator
- Compare with the Eguchi–Yang model:

$$\int_{H_N} \exp \operatorname{tr} N \left(V(X) + \sum_{\ell \geq 1} \frac{t_\ell}{\ell} X^\ell + 2 \sum_{\ell \geq 1} \widehat{t}_\ell X^\ell (\log X - c_\ell) \right) \mathrm{d}X$$

$$V(x) := -2x(\log x - 1), \quad c_\ell := \sum_{j=1}^{\ell} \frac{1}{j}.$$

Thank you for your attention!